Questions similar to what could be on the final:

Area, Volumes, Applications of Integrals

Compute the volume of the following solids:

1. $y = e^x, y = 1, x = 2$ rotated around $y = -2$.
2. $y = \sin(x), y = \cos(x), x = 0, y = 0, x = \frac{\sqrt{2}}{2}$ about $y = -1$.

Applications of integrals from 6.4:

1. (15) A cable that weighs $2kg/m$ is used to lift $800kg$ of coal up to a mine shaft which is $500m$ deep. Find the work done.
2. (9) Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. How much work is needed to stretch the spring from 35 cm to $40cm$? Hint: you may use Hooke's law, F=kx to compute k from the numbers given and then apply this to the question.
3. (17) A leaky 10 kg bucket is lifted from the ground to a height of 12 m at a constant speed with a rope that weighs 0.8 kg/m. Initially the bucket contains 36 kg of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 12 m level. How much work is done?
4. A tank with a capacity of 500 liters contains 500 liters of water with 100 kg of salt in solution. Water containing 1 kg of salt per liter is entering at the rate of 3 liters per minute, and the mixture is allowed to flow out of the tank at a rate of 3 liters per minute. Find the amount of salt in the tank after 5 minutes.

Solution: let $C(t)$ denote the amount of salt in the container at time t. Then from the text we read that $C(0) = 100kg$. Further, the amount $C(t)$ changes as follows:

1. per minute we are losing $\frac{C(t)}{500} \cdot 3$ kg of salt, since at a time t there are exactly $C(t)/500$ kg of salt in one liter of the solution
2. At the same time, we are adding water with a salt concentration of $1kg/l$ and 3 liters per minute, so we are adding 3 kg of salt per minute.

Hence we are adding 3 kg of salt per minute. This gives the following differential equation:

$$\frac{dC}{dt} = -\frac{3C}{500} + 3$$
$$\frac{dC}{dt} = \frac{-3C + 1500}{500}$$
$$\frac{dC}{-3C + 1500} = \frac{dt}{500}$$
$$\frac{dC}{3C - 1500} = -\frac{dt}{500}$$

Which leads to

$$C(t) = K \cdot e^{-\frac{3t}{500}} + 500.$$

Using the initial conditions $C(0) = 100$ we obtain $K = -400$ and so

$$C(t) = -400 \cdot e^{-\frac{3t}{500}} + 500.$$

Hence to answer the question of the amount of salt after 5 minutes, we evaluate $C(t)$ at $t = 5$: $C(5) = -400 \cdot e^{-\frac{15}{500}} + 500 = 111.82$ kg.

Improper Integrals

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\int_3^\infty \frac{1}{(x-2)^{3/2}} dx$.
2. $\int_1^5 \frac{1}{(x-2)^{2/3}} dx$.

Differential Equations

Use Euler's method to compute the following estimates:

1. Initial value problem $y' = y + 2x$ with $y(1) = 0$, take 4 approximation steps to estimate $y(2)$.
2. Step size $h = 0.1$ to estimate $y(0.3)$ of $y' = y + xy$ with $y(0) = 1$.

Solve the inital value problem:

1. $\frac{dy}{dx} = xy^2, y(0) = 3$.
2. $xy^2y' = x + 1, y(1) = 1$.
3. $\frac{dy}{dx} = xe^{-y}, y(2) = 1$.

Tests for convergence:

(a) $\sum_{n=1}^\infty n^2 e^{-n^3}$,

(b) $\sum_{n=2}^\infty \frac{1}{n(ln(n))^2}$,

(c) $\sum_{k=1}^\infty \frac{1}{k^2+k^3}$,

(d) $\sum_{i=1}^\infty \frac{i}{\sqrt{i^5+1}}$,

(e) $\sum_{s=1}^\infty \frac{s^2-5s}{s^3+s+1}$,

(f) $\sum_{i=1}^\infty \frac{ln(i)}{i}$,

(g) $\sum_{n=1}^\infty \frac{e^{1/n}}{n!}$,

(h) $\sum_{n=1}^\infty (-1)^{n+3} \frac{n^2}{n^3+4}$,

(i) $\sum_{n=1}^\infty (-1)^n \frac{\sqrt{n}}{2n+3}$,

(j) $\sum_{n=1}^\infty \frac{n!}{100^n}$

(k) $\sum_{n=1}^\infty \frac{(-2)^n}{n^n}$

(l) $\sum_{n=1}^\infty (-1)^{n-1} \frac{n}{n^2+4}$

Series arithmetic

1. Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_{10} as an approximation to the sum of the series.
2. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ up to three decimal places. Remark: the calculation of the final sum may be longer than what will be asked in an exam.
3. Harder: Determine the sums of $\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!}$, $\sum_{n=0}^{\infty} \frac{3^n}{2^n n!}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{2^{2n}}$. Hint: some may look like MacLaurin series of functions that you should remember. For the last series: try to split it into a positive and negative part to get two series that you recognise.

Find radius and Interval of Convergence:

1. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$
2. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$
3. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$
4. $\sum_{n=1}^{\infty} \frac{n \cdot (x+1)^n}{4^n}$

Taylor and MacLaurin series

Give (quoting from textbook allowed!) the MacLaurin series and their radii of convergence for:

1. $\frac{1}{1-x}$
2. e^x
3. $\sin x$
4. $\cos x$
5. $\arctan x$
6. $\ln(1+x)$
7. $(1+x)^k$

Compute the MacLaurin series (by using the definition OR by subsituting into a known series) for the following functions:

1. e^{-2x^2}
2. $x \cdot \cos(x)$
3. $\cos(x^2)$
4. $x \cdot \cos(\frac{1}{2}x^2)$

Use the MacLaurin series expansions to compute the following integrals:

1. $\int \sin(x^2)dx$
2. $\int \ln(1+x^2)dx$
3. $\int e^{3\sqrt{x}}dx$

What is the general formula of the MacLaurin series? What is the general forumla for the Taylor Series at a point a?

Functions in two variables

Sketch the graph of the following functions. Start with drawing a few contour lines (=level curves) and the graphs for setting $y = k$ or $x = k$ for several values of k. Find the tangent plane equations at the given points and the linear approximations of the functions. If you are unsure about the shape of the contour lines, use a table of values.

Further, determine the direction of the gradient vector at a few points in the sketch of the contour lines.

1. Trickier: $f(x, y) = y^2 - 3xy + 3x^2$, tangent plane at $(1, 2)$. Hint: For the contour lines you get an equation $k = y^2 - 3xy + 3x^2$. By substituting $u = y - \frac{3}{2}x$ you can see that it becomes $k = u^2 + 3x^2$.
2. $f(x, y) = \frac{3x-y}{x}$, tangent plane at $(3, 4)$.
3. $f(x, y) = \sqrt{4x^2 + y^2}$, tangent plane at $(1, 1)$.
4. $f(x, y) = ye^x$, tangent plane at $(0, 0)$.
5. $f(x, y) = \ln(x^2 + 4y^2)$, tangent plane at $(1, 0)$.

I highly recommend looking through exercises 59-64 of 14.1.

Partial Derivatives

Find all second partial derivatives of f at the give points.

1. $f(x, y) = xy \sin(y^2)$ at $(3, 2)$.
2. $f(x, y) = y^5 - 3x^2y$ at $(4, 1)$.
3. $f(x, y) = \frac{x}{(x^2+y)^2}$ at $(1, 1)$.
4. $f(x, y) = \sqrt{x} \ln(y^x)$ at $(1, 4)$.

1. Find $f_{zyx} = \frac{\partial f}{\partial x \partial y \partial z}$ of $f(x, y, z) = 3xyz + x^2y^3z^7$. Evaluate at $(1, 1, 3)$.
2. Find $f_{yyx} = \frac{\partial f}{\partial x \partial y \partial y}$ of $f(x, y, z) = x^{\frac{12}{y}} + \ln(xy) + \sqrt{yz}$. Evaluate at $(1, 2, 3)$.

Use the chain rule to compute the following:

1. $z = x^4 + x^2y$, $x = s + 2t$, $y = st$
2. $f(x, y) = \frac{x+y}{x-y}$, $x = 3st^2$, $y = s^2 + t^2$

Directional Derivatives

Beware: the directional vector may need to be of a certain length.

1. $f(x, y) = e^x \sin(y)$, $u = \langle 1, 5 \rangle$ at $(0, \frac{\pi}{4})$
2. $f(x, y) = \frac{x}{x^2+y^2}$, approach from angle $\theta = \pi/3$, evaluate at $(2, 4)$
3. $f(x, y) = x^2y - xy$ from $u = 3 \cdot \mathbf{i} + 6 \cdot \mathbf{j}$. Recall that $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Good luck with studying!